

# STRUCTURAL CONDITIONS FOR FULL MHD EQUATIONS

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**ABSTRACT.** In this paper, we investigate the characteristic structure of the full equations of magnetohydrodynamics (MHD) and show that it satisfies the hypotheses of a general variable-multiplicity stability framework introduced by Métivier and Zumbrun, thereby extending to the general case various results obtained by Métivier and Zumbrun for the isentropic equations of MHD.

## 1. INTRODUCTION

Stability of hyperbolic boundary value problems and the related shock problems has been studied by a number of authors for years. There are satisfactory theories for symmetric systems with dissipative boundary conditions or hyperbolic systems with constant multiplicities provided that the boundary conditions satisfy Lopatinski conditions. Many applications fall into one of these two cases. But the equations of magnetohydrodynamics (MHD) fail to satisfy the assumption of constant multiplicities. Moreover, though they are symmetrizable, boundary conditions arising in the shock stability problem are not dissipative. This motivates the further study of the variable multiplicities case. It is proved by [MeZ] for symmetrizable systems that under certain conditions on eigenvalues, the uniform Lopatinski condition is equivalent to the maximal energy estimate which gives linearized and nonlinear stability. Specifically, they extend the construction of Kreiss' symmetrizers to the case that eigenvalues are not geometrically regular provided that they are "totally nonglancing" or "linearly splitting" in a sense that they define. (See section 3 for their definitions) This yields the maximal estimate when the uniform Lopatinski condition is satisfied. In [MeZ], it is also showed that all characteristics of isentropic MHD are either "geometrically regular" or "totally nonglancing". Hence, a satisfactory framework for the study of linearized and nonlinear stability of isentropic MHD is achieved, through which various results are obtained. The goal of this paper is to investigate the eigenvalues of full MHD equations. It turns out that all characteristics are algebraically regular. More specifically, they are either geometrically regular or totally nonglancing as the isentropic case.

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As a consequence, all results obtained in [MeZ] for the isentropic case extend immediately to the full MHD equations.

## 2. DEFINITIONS

Consider a first-order system

$$(2.1) \quad L(p, \partial_t, \partial_x)U = U_t + \sum_{j=1}^d A^j(p)U_{x_j},$$

and its symbolic operator

$$(2.2) \quad \tilde{L}(p, \tau, \xi) = \tau Id + \sum_{j=1}^d \xi_j A^j(p).$$

The characteristic polynomial of (2.2) is  $P(p, \tau, \xi) := \det \tilde{L}(p, \tau, \xi)$ .

**Definition 2.1.** (D1) *The system (2.1) is hyperbolic if  $\sum_{j=1}^d \xi_j A_j(p)$  has all real, semisimple eigenvalues for all  $\xi \in \mathbb{R}^d$ .*

(D2) *The system (2.1) is symmetric hyperbolic in the sense of Friedrichs if there is a positive definite smooth symmetrizer  $S(p)$  such that  $S(p)A_j(p)$  is symmetric for all  $j = 1, \dots, d$ .*

Let  $(p_0, \tau_0, \xi_0)$  be a root of  $P(p, \tau, \xi) = 0$  whose multiplicity is  $m$  in  $\tau$ .

**Definition 2.2.** (a)  *$(p_0, \tau_0, \xi_0)$  is algebraically regular if on a neighborhood  $\omega$  of  $(p_0, \xi_0)$ , there are  $m$  smooth real functions  $\lambda_j(p, \xi)$ , analytic in  $\xi$ , such that for  $(p, \xi)$ ,*

$$(2.3) \quad P(p, \tau, \xi) = e(p, \tau, \xi) \prod_{j=1}^m (\tau + \lambda_j(p, \xi))$$

*where  $e$  is a polynomial in  $\tau$  and smooth coefficients in  $\xi$  and  $p$  such that  $e(p_0, \tau_0, \xi_0) \neq 0$ .*

(b)  *$(p_0, \tau_0, \xi_0)$  is geometrically regular if in addition there are  $m$  smooth vectors  $v_j(p, \xi)$ , analytic in  $\xi$ , such that*

$$(2.4) \quad A(p, \xi)v_j(p, \xi) = \lambda_j(p, \xi)v_j(p, \xi)$$

*where  $v_1, \dots, v_m$  are linearly independent.*

**Definition 2.3.** *A root  $(p_0, \tau_0, \eta_0, \xi_0)$  of  $P$ , of multiplicity  $m$  in  $\tau$  is nonglancing if the  $m$ th order of Taylor expansion of  $P$  at  $(p_0, \tau_0, \eta_0, \xi_0)$  satisfies*

$$(2.5) \quad P^m(dx) \neq 0$$

*where  $dx$  is the (space-time) conormal to the boundary. Moreover, it is called totally nonglancing if it is either totally incoming or outgoing.*

**Assumption 2.4.** *Suppose that  $L$  is symmetric hyperbolic and noncharacteristic with respect to the boundary  $x_d = 0$ , and that its characteristic roots are geometrically regular or totally nonglancing.*

Consider a hyperbolic system

$$(2.6) \quad L(p, \partial_t, \partial_{\bar{x}}, \partial_{x_d}) = A_d(p)(\partial_{x_d} + G(p, \partial_t, \partial_{\bar{x}}))$$

with the assumption that the boundary  $\{x_d = 0\}$  is noncharacteristic, i.e.,  $\det A_d(p) \neq 0$ .

The classical plane wave analysis yields the boundary value problems

$$(2.7) \quad \begin{cases} L(p, \zeta) \tilde{U} = \partial_{x_d} \tilde{U} + iG(p, \zeta) \tilde{U} = f, \\ M(p, \zeta) \tilde{U}|_{x_d=0} = g, \end{cases}$$

where

$$(2.8) \quad G(p, \zeta) = A_d^{-1}(p)((\tau - i\gamma)Id + \sum_{j=1}^{d-1} \eta_j A^j(p))$$

with a new variable  $\zeta = (\tau - i\gamma, \eta)$ .

Let  $\mathbf{E}_-(p, \zeta)$  be the invariant subspace of  $G(p, \zeta)$  associated to the eigenvalues in  $\{\operatorname{Im} \mu < 0\}$ .

**Definition 2.5.** *The Lopatinski determinant associated with  $(L, M)(p, \zeta)$  is defined by*

$$(2.9) \quad D(p, \zeta) := \det(\mathbf{E}_-(p, \zeta), \ker M(p, \zeta)).$$

*We say that  $(L, M)(p, \zeta)$  satisfies the uniform Lopatinski condition on a neighborhood  $\omega$  if there is a constant  $c > 0$  such that*

$$(2.10) \quad |D(p, \zeta)| \geq c \text{ for } \forall (p, \zeta) \in \omega \times S_+^d.$$

### 3. COMPUTATION

The inviscid full MHD Equations are

$$(3.1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho(u_t + u \cdot \nabla u) + \nabla P - \operatorname{curl} B \times B = 0, \\ \rho e_\theta(\theta_t + u \cdot \nabla \theta) + \theta P_\theta \operatorname{div} u = 0, \\ B_t + \operatorname{curl}(B \times u) = 0 \end{cases}$$

with the constraint

$$(3.2) \quad \operatorname{div} B = 0,$$

where  $\rho \in \mathbb{R}$  is density,  $u \in \mathbb{R}^3$  is velocity,  $P(\rho, \theta) \in \mathbb{R}$  is pressure,  $B \in \mathbb{R}^3$  is magnetic field and  $\theta \in \mathbb{R}$  is temperature.

Using the identity  $\operatorname{curl}(B \times u) = (\operatorname{div} u)B + u \cdot \nabla B - B \cdot \nabla u - (\operatorname{div} B)u$  and the constraint (3.2), (3.1) can be put in the symmetric form

$$(3.3) \quad \begin{cases} \rho_t + u \cdot \nabla \rho + \rho \operatorname{div} u = 0, \\ u_t + u \cdot \nabla u + \rho^{-1}(P_\rho \nabla \rho + P_\theta \nabla \theta) - \rho^{-1}(\operatorname{curl} B \times B) = 0, \\ \theta_t + u \cdot \nabla \theta + \frac{\theta P_\theta}{\rho e_\theta} \operatorname{div} u = 0, \\ B_t + u \cdot \nabla B + (\operatorname{div} u)B - B \cdot \nabla u = 0. \end{cases}$$

Linearizing (3.3) about  $(\rho, u, \theta, B)$ , we have the linearized equations

$$(3.4) \quad \begin{cases} D_t \dot{\rho} + \rho \operatorname{div} \dot{u} = 0, \\ D_t \dot{u} + \rho^{-1}(P_\rho \nabla \dot{\rho} + P_\theta \nabla \dot{\theta}) - \rho^{-1}(\operatorname{curl} \dot{B} \times B) = 0, \\ D_t \dot{\theta} + \frac{\theta P_\theta}{\rho e_\theta} \operatorname{div} \dot{u} = 0, \\ D_t \dot{B} + (\operatorname{div} \dot{u})B - B \cdot \nabla \dot{u} = 0 \end{cases}$$

where  $D_t = \partial_t + u \cdot \nabla$  is the material derivative.

By taking Fourier Transform in  $x$  variable, we have the symbol matrix associated to (3.4)

(3.5)

$$\begin{aligned} \tilde{A}(U, \xi) &:= (-u \cdot \xi)I + \sum_j A^j(U) \xi_j \\ &= \begin{pmatrix} 0 & \rho \xi & 0 & 0 \\ \rho^{-1} P_\rho \xi^{tr} & 0 & \rho^{-1} P_\theta \xi^{tr} & \rho^{-1}(\xi^{tr} B - (B \cdot \xi)I) \\ 0 & \rho^{-1} P_\theta \theta / e_\theta \xi & 0 & 0 \\ 0 & B^{tr} \xi - (B \cdot \xi)I & 0 & 0 \end{pmatrix}. \end{aligned}$$

So, we can express the linear part of (3.4) in the matrix form

(3.6)

$$\tilde{A}(U, \xi) \begin{pmatrix} \dot{\rho} \\ \dot{u} \\ \dot{\theta} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} \rho(\xi \cdot \dot{u}) \\ \rho^{-1} P_\rho \dot{\rho} \xi^{tr} + \rho^{-1} P_\theta \dot{\theta} \xi^{tr} + \rho^{-1}(B \cdot \dot{B}) \xi^t - \rho^{-1}(B \cdot \xi) \dot{B}^t \\ \rho^{-1} P_\theta \theta / e_\theta (\xi \cdot \dot{u}) \\ B^t(\xi \cdot \dot{u}) - (B \cdot \xi) \dot{u}^t \end{pmatrix}.$$

Now, consider the eigenvalue equation associated to (3.4)

$$(3.7) \quad \tilde{A}(U, \xi) \begin{pmatrix} \dot{\rho} \\ \dot{u} \\ \dot{\theta} \\ \dot{B} \end{pmatrix} = \lambda \begin{pmatrix} \dot{\rho} \\ \dot{u} \\ \dot{\theta} \\ \dot{B} \end{pmatrix}.$$

Equivalently, we have

$$(3.8) \quad \begin{cases} \rho(\xi \cdot \dot{u}) = \lambda \dot{\rho}, \\ \rho^{-1} P_\rho \dot{\rho} \xi^t + \rho^{-1} P_\theta \dot{\theta} \xi^t + \rho^{-1} (B \cdot \dot{B}) \xi^t - \rho^{-1} (B \cdot \xi) \dot{B}^t = \lambda \dot{u}^t, \\ \rho^{-1} P_\theta \theta / e_\theta (\xi \cdot \dot{u}) = \lambda \dot{\theta}, \\ (\xi \cdot \dot{u}) B^t - (B \cdot \xi) \dot{u}^t = \lambda \dot{B}. \end{cases}$$

We adopt the variable  $\xi$  for the spatial frequencies and let

$$(3.9) \quad \xi = |\xi| \hat{\xi}, \quad u_\parallel = \hat{\xi} \cdot u, \quad u_\perp = u - u_\parallel \hat{\xi}.$$

Then we can rewrite the eigenvalue equations (3.8) as

$$(3.10) \quad \begin{cases} \rho \dot{u}_\parallel = \tilde{\lambda} \dot{\rho}, \\ P_\rho \dot{\rho} + P_\theta \dot{\theta} + B_\perp \cdot \dot{B}_\perp = \tilde{\lambda} \rho \dot{u}_\parallel, \\ -\mu_0^{-1} B_\parallel \dot{B}_\perp = \tilde{\lambda} \rho \dot{u}_\perp, \\ P_\theta \theta / e_\theta \dot{u}_\parallel = \rho \tilde{\lambda} \dot{\theta}, \\ 0 = \tilde{\lambda} \dot{B}_\parallel, \\ \dot{u}_\parallel B_\perp - B_\parallel \dot{u}_\perp = \tilde{\lambda} \dot{B}_\perp. \end{cases}$$

Let  $\dot{\sigma} := \dot{\rho}/\rho$  and  $\dot{v} := \dot{B}/\sqrt{\rho}$ . The characteristic system (3.10) reads

$$(3.11) \quad \begin{cases} \dot{u}_\parallel = \tilde{\lambda} \dot{\sigma}, \\ P_\rho \dot{\sigma} + \rho^{-1} P_\theta \dot{\theta} + v_\perp \cdot \dot{v}_\perp = \tilde{\lambda} \dot{u}_\parallel, \\ -v_\parallel \dot{v}_\perp = \tilde{\lambda} \dot{u}_\perp, \\ \frac{P_\theta \theta}{e_\theta \rho} \dot{u}_\parallel = \tilde{\lambda} \dot{\theta}, \\ 0 = \tilde{\lambda} \dot{v}_\parallel, \\ v_\perp \dot{u}_\parallel - v_\parallel \dot{u}_\perp = \tilde{\lambda} \dot{v}_\perp. \end{cases}$$

Now, we introduce new variables

$$(3.12) \quad \begin{cases} \dot{x} := P_\rho \dot{\sigma} + \rho^{-1} P_\theta \dot{\theta}, \\ \dot{y} := P_\theta \theta \dot{\sigma} - e_\theta \rho \dot{\theta}. \end{cases}$$

Note that the transformation  $(\dot{\sigma}, \dot{\theta}) \mapsto (\dot{x}, \dot{y})$  is invertible if and only if  $\rho P_\rho e_\theta + \rho^{-1} \theta P_\theta^2 \neq 0$ . So, it is invertible since  $P_\rho + \frac{P_\theta^2 \theta}{\rho^2 e_\theta} > 0$ . Using new

variables  $(\dot{x}, \dot{y})$ , we have the equivalent system

$$(3.13) \quad \begin{cases} (P_\rho + \frac{P_\theta^2 \theta}{\rho^2 e_\theta}) \dot{u}_\parallel = \tilde{\lambda} \dot{x}, \\ 0 = \tilde{\lambda} \dot{y}, \\ \dot{x} + v_\perp \cdot \dot{v}_\perp = \tilde{\lambda} \dot{u}_\parallel, \\ -v_\parallel \dot{v}_\perp = \tilde{\lambda} \dot{u}_\perp, \\ 0 = \tilde{\lambda} \dot{v}_\parallel, \\ v_\perp \dot{u}_\parallel - v_\parallel \dot{u}_\perp = \tilde{\lambda} \dot{v}_\perp. \end{cases}$$

Note that the zero-eigenspace of  $\tilde{A}$  is  $\mathbb{E}_0 := \{\dot{x} = 0, \dot{u} = 0, \dot{v}_\perp = 0\}$ , which is of dimension 2. On the space  $\mathbb{E}_0$ ,  $\tilde{A} = 0$ . This implies  $A = u \cdot \xi$  on  $\mathbb{E}_0$ . Thus,  $u \cdot \xi$  is an eigenvalue of  $A$  of constant multiplicity 2. By inspection, it is both algebraically and geometrically regular.

Taking a basis of  $\xi^\perp$  such that  $v_\perp = (b, 0)$  and letting  $a = v_\parallel$ , we can write (3.13) in the coordinates of  $(\dot{x}, \dot{u}_\parallel, \dot{u}_\perp^1, \dot{u}_\perp^2, \dot{v}_\perp^1, \dot{v}_\perp^2, \dot{y}, \dot{v}_\parallel)^t$  as follows:

$$(3.14) \quad -\tilde{A}_0 + \tilde{\lambda} I = \begin{pmatrix} \tilde{\lambda} & -\gamma^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \tilde{\lambda} & 0 & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & \tilde{\lambda} & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{\lambda} & 0 & a & 0 & 0 \\ 0 & -b & a & 0 & \tilde{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & \tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\lambda} \end{pmatrix} = 0$$

where  $\gamma^2 = P_\rho + \frac{P_\theta^2 \theta}{\rho^2 e_\theta} > 0$ .

So, the associated characteristic polynomial  $P(x)$  is

$$(3.15) \quad \begin{aligned} P(x) &= x^2(x^2 - a^2)((x^2 - a^2)(x^2 - (P_\rho + \frac{P_\theta^2 \theta}{\rho^2 e_\theta})) - b^2 x^2) \\ &= x^2(x^2 - a^2)((x^2 - a^2)(x^2 - \gamma^2) - b^2 x^2). \end{aligned}$$

Thus,

$$(3.16) \quad \begin{cases} \tilde{\lambda}^2 = 0, \\ \tilde{\lambda}^2 = a^2, \\ \tilde{\lambda}^2 = c_f^2 := \frac{1}{2}((\gamma^2 + h^2) + \sqrt{(\gamma^2 - h^2)^2 + 4b^2 \gamma^2}), \\ \tilde{\lambda}^2 = c_s^2 := \frac{1}{2}((\gamma^2 + h^2) - \sqrt{(\gamma^2 - h^2)^2 + 4b^2 \gamma^2}) \end{cases}$$

where  $h^2 = a^2 + b^2 = |B|^2/\rho$ .

**Remark 3.1.** *Coordinate change (3.12) is the key step in the computation, leading to block decomposition (3.14).*

#### 4. RESULTS

In the analysis of eigenvalues of (3.1), what is left is the same as the corresponding steps carried out in [MeZ] for the isentropic case. Specifically, in the isentropic case treated in [MeZ], the matrix corresponding to (3.14) has the same form, with identical  $6 \times 6$  upper lefthand block and lower lefthand block of the same form  $\tilde{\lambda}I_k$ , but with dimension  $k = 1$  instead of  $k = 2$  as in the present case. Thus, decomposing into these two blocks, all the analysis goes as before. We summarize for completeness.

**Lemma 4.1.** *Let  $\gamma^2 = P_\rho + \frac{P_\theta^2 \theta}{\rho^2 e_\theta} > 0$ . The 8 eigenvalues are*

$$(4.1) \quad \begin{cases} \lambda_0 = \xi \cdot u \text{ (of multiplicity 2),} \\ \lambda_{\pm 1} = \xi \cdot u \pm c_s |\xi|, \\ \lambda_{\pm 2} = \xi \cdot u \pm (\xi \cdot B)/\sqrt{\rho}, \\ \lambda_{\pm 3} = \xi \cdot u \pm c_f |\xi| \end{cases}$$

where

$$(4.2) \quad c_f = \frac{1}{2}((\gamma^2 + h^2) + \sqrt{(\gamma^2 - h^2)^2 + 4b^2\gamma^2}),$$

$$(4.3) \quad c_s = \frac{1}{2}((\gamma^2 + h^2) - \sqrt{(\gamma^2 - h^2)^2 + 4b^2\gamma^2}),$$

$$(4.4) \quad h^2 = |B|^2/\rho,$$

and

$$(4.5) \quad b^2 = |\hat{\xi} \times B|^2/\rho.$$

**Lemma 4.2.** *Assume that  $0 < |B|^2 \neq \rho\gamma^2$  where  $\gamma^2 = P_\rho + \frac{P_\theta^2 \theta}{\rho^2 e_\theta} > 0$ .*

(a) *When  $\xi \cdot B \neq 0$  and  $\xi \times B \neq 0$ , the eigenvalues  $\lambda_{\pm 1}, \lambda_{\pm 2}$  and  $\lambda_{\pm 3}$  are simple and  $\lambda_0$  is semisimple of constant multiplicity 2 (geometrically regular).*

(b) *On the manifold  $\xi \cdot B = 0, \xi \neq 0$ , the eigenvalues  $\lambda_{\pm 3}$  are simple and the multiple eigenvalues  $\lambda_0 = \lambda_{\pm 1} = \lambda_{\pm 2}$  are geometrically regular.*

(c) *On the manifold  $\xi \times B = 0, \xi \neq 0$ , the multiple eigenvalue  $\lambda_0$  is semisimple. When  $|B|^2 < \rho\gamma^2$  (resp.  $|B|^2 > \rho\gamma^2$ ),  $\lambda_{\pm 3}$  (resp.  $\lambda_{\pm 1}$ ) are simple and  $\lambda_{+2} = \lambda_{+1}$  (resp.  $\lambda_{+3}$ )  $\neq \lambda_{-2} = \lambda_{-1}$  (resp.  $\lambda_{-3}$ ) are double and algebraically regular, and not geometrically regular (totally nonglancing, provided that  $u_3 - \sigma \neq \pm B_3/\sqrt{\rho}$ ).*

*Proof.* The characteristic polynomial is the same as the one for the isentropic case, except additional factor  $x$ . The zero-eigenspace is of dimension 2, independent of  $\xi$ . Thus, the eigenvalue  $\lambda_0 = \xi \cdot u$  is geometrically regular. For other eigenvalues, the result follows from the calculation in [MeZ].  $\square$

**Corollary 4.3.** *For the linearized equations of (full) MHD, the uniform Lopatinski condition is equivalent to the uniform stability estimate (implying linearized and nonlinear stability).*

**Corollary 4.4.** *Under the generically satisfied conditions  $0 < |B^\pm|^2 \neq \rho^\pm(\gamma^\pm)^2$ , the uniform Lopatinski condition implies linearized and nonlinear stability of noncharacteristic Lax-type MHD shocks.*

As an application, we recover for the full equations of MHD the result of Blokhin and Trakhinin that MHD shocks are stable in the small-magnetic field limit  $B \rightarrow 0$ .

**Corollary 4.5.** *In the  $B \rightarrow 0$  limit, Lax-type MHD shocks approaching a noncharacteristic limiting fluid-dynamical shock satisfy the Lopatinski condition if and only if it is satisfied by the limiting fluid-dynamical shock (i.e., the shock satisfies the physical stability conditions of Erpenbeck–Majda [Er, Maj], in which case the MHD shocks are linearly and nonlinearly stable. In particular, for an ideal gas equation of state, they are always stable in the  $B \rightarrow 0$  limit.*

**Remark 4.6.** Verification of the uniform Lopatinski condition has been carried out numerically or analytically for several interesting cases; see [BT1, BT2].

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